Eur. Phys. J. C 17, 407–411 (2000)
Digital Object Identifier (DOI) 10.1007/s100520000478

THE EUROPEAN PHYSICAL JOURNAL C

© Società Italiana di Fisica
Springer-Verlag 2000

Parameter counting for neutrino mixing

Yu.F. Pirogov[1,2]

[1] Institute for High Energy Physics, Protvino, 142284 Moscow Region, Russia
[2] Moscow Institute of Physics and Technology, Dolgoprudny, Moscow Region, Russia

Received: 25 April 2000 / Revised version: 16 June 2000 /
Published online: 8 September 2000 – © Springer-Verlag 2000

Abstract. The content of physical masses, mixing angles and CP violating phases in the lepton sector of the extended standard model, both renormalizable and non-renormalizable, with arbitrary numbers of singlet and left-handed doublet neutrinos is systematically analyzed in the weak basis.

1 Introduction

Quark mixing in the minimal standard model (SM) of the strong and electroweak interactions is nowadays well understood. It is described by the Cabibbo–Kobayashi–Maskawa (CKM) unitary mixing matrix [1] for the quark charged currents, the neutral ones and Yukawa interactions being flavor conserving. As for the lepton sector, the SM exhibits an extremely simple and economic structure. It counts just three physical parameters, the charged lepton masses, and predicts no flavor and CP violation. But it has been widely recognized that the inclusion of the (iso)singlet neutrinos and/or neutrino masses in the SM would result in lepton mixing and flavor violation, with all such related phenomena as neutrino oscillations [2], CP violation, etc. (for a recent review see, e.g., [3]).

There are two principal differences between the lepton and quark mixings. First, the number of singlet neutrinos relative to that of the (iso)doublet ones is not restricted by the chiral anomalies and hence can be arbitrary. Second, Majorana masses for neutrinos are possible in addition to the Dirac ones. This inevitably complicates the proper SM extensions and proliferates the free parameters. Hence, the immediate problem arises of how to extract the physical parameters, to separate the masses, mixing angles and CP violating phases among them, as well as to conveniently parameterize the mixing matrices. There have been many studies of these related topics. The case with an arbitrary number of left-handed doublet neutrinos but without singlet ones was considered in [4]; the case with equal arbitrary numbers of singlet and doublet neutrinos in [5], and the general case with arbitrary numbers of both types of neutrinos in [6]. In particular, the last case with only Dirac masses under the condition that the number of singlet neutrinos is not higher than that of the doublet ones was studied in [7]. Traditionally, these investigations were carried out by an explicit construction in the mass basis.

An alternative approach to parameter counting is also feasible. It can be formulated in the weak basis entirely through the symmetry properties of a model before its spontaneous symmetry breaking [8]. As for the lepton sector, this approach was applied in [8] to the n family renormalizable SM with one right-handed neutrino per family. In the present paper, it is extended to the general case. In this way, all the possible parameter space configurations of the SM with in addition arbitrary numbers of singlet and left-handed doublet neutrinos are systematically analyzed. Both renormalizable and non-renormalizable extensions of the SM, among them the pure Dirac and pure Majorana cases, are considered. In a consistent fashion, the known results on lepton parameter counting are recovered, some of them having been corrected. Moreover, new ones are obtained. The relation between the weak and mass basis countings is clarified. The results on parameter counting for neutrino mixing are summarized in the tables.[1]

2 Renormalizable extensions

2.1 Arbitrary case

The most general renormalizable $SU(2)_W \times U(1)_Y$ invariant lepton Lagrangian of the SM including right-handed neutrinos reads

$$\mathcal{L} = \overline{l_{\rm L}}\mathrm{i}\not{D} l_{\rm L} + \overline{e_{\rm R}}\mathrm{i}\not{D} e_{\rm R} + \overline{\nu_{\rm R}}\mathrm{i}\not{\partial}\nu_{\rm R} - \left(\overline{l_{\rm L}}Y^e e_{\rm R}\phi + \overline{l_{\rm L}}Y^\nu \nu_{\rm R}\phi^C + \frac{1}{2}\overline{\nu_{\rm L}^C}M\nu_{\rm R} + \text{h.c.}\right). \quad (1)$$

Here the lepton doublet $l_{\rm L}$ and singlet $e_{\rm R}$, $\nu_{\rm R}$ fields mean those in a weak basis where, by definition, the symmetry

[1] In fact, what we are talking about is lepton mixing which is described by a counterpart of the CKM matrix. But one can always choose a weak basis where the mixing matrix of charged leptons is unity. In this sense, lepton mixing is synonymous with neutrino mixing.

Table 1. Parameter counting for the SM renormalizable extension $(d, s)_{\rm r}$ with d doublet neutrinos and s singlet ones. In this table and the ones which follow, the first and the second groups of moduli for the physical mass matrix $\mathcal{M}_{\rm ph}$ correspond to the independent mixing angles and masses, respectively

Couplings and symmetries	Moduli	Phases
Y^e, Y^ν, M	$d^2 + ds + s(s+1)/2$	$d^2 + ds + s(s+1)/2$
$G = U(d)^2 \times U(s)$	$-d(d-1) - s(s-1)/2$	$-d(d+1) - s(s+1)/2$
$H = I$	0	0
$\mathcal{M}_{\rm ph}(d,s)_{\rm r}$	$sd + (d+s)$	$d(s-1)$
$\mathcal{M}_{\rm ph}(n,n)_{\rm r}$	$n^2 + 2n$	$n(n-1)$

properties are well stated. It is supposed that the ordinary chiral families of the SM with the doublet left-handed Weyl neutrinos with number $d \ge 3$ are added by the singlet Weyl neutrinos with number $s \ge 0$. Let us designate the SM extended in such a renormalizable manner as the $(d, s)_{\rm r}$ extension. A priori, one should retain s and d as arbitrary integers, both $s \le d$ and $s > d$ being allowed. In the present analysis, we omit possible vector-like lepton doublets. Hence, taking in account the most probable exclusion of the fourth heavy chiral family [9], one should actually put $d = 3$. Nevertheless, we retain d as a free parameter to better elucidate the parameter space structure of the extended SM. Furthermore, $\not{D} \equiv \gamma^\alpha D_\alpha$ is the generic covariant derivative which reduces to the ordinary one, $\not{\partial} = \gamma^\alpha \partial_\alpha$, for the hypercharge zero singlet neutrinos. Here and in what follows, the notation $\nu_{\rm L}^C \equiv (\nu_{\rm R})^C = C\overline{\nu_{\rm R}}^{\rm T}$, etc. is used for the particle–antiparticle conjugates of the chiral fermions. Y^e and Y^ν are the arbitrary complex $d \times d$ and $d \times s$ Yukawa matrices, respectively, and M is the complex symmetric $s \times s$ matrix of the Majorana masses for the singlet neutrinos. Finally, ϕ is the Higgs isodoublet and $\phi^C \equiv {\rm i}\tau_2 \phi^*$ is its charge conjugate.

The parameter counting in the weak basis for the lepton sector of the extended SM proceeds as is shown in Table 1. Here G is the global symmetry of the kinetic part of the Lagrangian (1). Due to the Dirac and Majorana mass terms the symmetry G is explicitly violated so that the residual symmetry is trivial, $H = I$. In what follows, we generally assume that there are no mass textures or that there is no accidental mass degeneracy. Otherwise, the residual symmetry would increase, and special consideration of each particular case would be mandatory. The transformations of the broken part G/H (here $G/H = G$) can be used to absorb the spurious parameters in (1), leaving only the independent physical set, $\mathcal{M}_{\rm ph}$, of them. For this reason, parameters corresponding to the symmetry G are represented in the tables with a minus sign, whereas those of H are denoted with a plus sign. As a result, $\mathcal{M}_{\rm ph}$ contains $sd + d + s$ independent moduli and $d(s-1)$ phases[2]. In all this, a real $\mathcal{M}_{\rm ph}$ corresponds to CP conservation.

[2] Note that our counting for the renormalizable $(d, s)_{\rm r}$ extension, both at $s \le d$ and $s > d$, disagrees with that in [6] (see remarks in Sect. 4)

We should stress that the weak basis counting fixes only the number of *independent* physical moduli (as well as the phases), generally underestimating the number of *actual* physical moduli. The reason is that, because of the absence of the left-handed Majorana masses, there are relations in the $(d, s)_{\rm r}$ extension between the actual mixing angles and masses. Considering all the masses as independent ones while a part of the mixing angles is taken as a function of the masses, would result in the superficial number of mixing angles being less than their actual number. This may cause some confusion in an explicit parametrization. So, it is more instructive to choose all the mixing angles as independent ones, considering part of the masses as a function of the angles and the rest of the masses. To decide what is the minimal number of independent masses, consider the limit $M \to \infty$ corresponding to decoupling of s heavy Majorana neutrinos. In this limit, the rest of the d Majorana neutrinos should necessarily become massless. Thus d Majorana masses depend on s ones. Clearly, it is impossible to further reduce the number of independent masses.

Finally, of the independent physical moduli, sd ones are mixing angles, the rest being masses of d charged leptons and s Majorana neutrinos. At $0 < s \le d$, there additionally appear s induced Majorana masses, $d - s$ neutrinos still remaining massless. This reflects the fact that in this case the rank of the neutrino mass matrix is $2s$. At $s > d > 0$, all $d + s$ Majorana neutrinos acquire masses. We stress that in Table 1 and the tables which follow, the number of physical masses chosen as independent ones is collectively that for both the charged leptons and neutrinos (Majorana or Dirac, depending on the context). The last line in Table 1 illustrates the extended n family SM with one right-handed neutrino per family[3]. It is clear that in contrast to the quark sector, the CP violation would generally take place at more than one complete lepton family.

In the SM-like case $(d, 0)_{\rm r}$ one has $G = U(d)^2$, all the neutrinos are massless and the residual symmetry in-

[3] As for division of the physical moduli into mixing angles and masses, a superficial disagreement for this case with [8] is caused by the fact that in that paper all the masses are supposed to be independent ones, while we, as is stated above, take as independent all the mixing angles

Table 2. Parameter counting for the SM renormalizable extension $(d, s)_{\rm D}$ with only Dirac masses. The number of physical masses in $\mathcal{M}_{\rm ph}$ is that of the Dirac ones

Couplings and symmetries	Moduli	Phases
Y^e, Y^ν	$d^2 + ds$	$d^2 + ds$
$G = U(d)^2 \times U(s)$	$-d(d-1) - s(s-1)/2$	$-d(d+1) - s(s+1)/2$
$H = U(1), 0 < s \le d$	0	1
$H = U(s-d) \times U(1)$ $0 < d < s$	$(s-d)(s-d-1)/2$	$(s-d)(s-d+1)/2 + 1$
$\mathcal{M}_{\rm ph}(d,s)_{\rm D}, 0 < s \le d$	$s(2d-s-1)/2 + (d+s)$	$s(2d-s-1)/2 - d + 1$
$\mathcal{M}_{\rm ph}(d,s)_{\rm D}, 0 < d < s$	$d(d-1)/2 + 2d$	$(d-1)(d-2)/2$
$\mathcal{M}_{\rm ph}(n,n)_{\rm D}$, $n > 0$	$n(n-1)/2 + 2n$	$(n-1)(n-2)/2$

creases up to $H = U(1)^d$ of the individual lepton numbers. Hence both the number of mixing angles and that of the phases are equal to zero, as it should be.

2.2 Only Dirac masses

There is an important case of SM extension which is renormalizable. Namely, the lepton number conservation would forbid Majorana mass terms, both left- and right-handed ones. In the absence of these masses the residual symmetry at $0 < s \le d$ would increase up to $H = U(1)$ of the total lepton number. In this case – designate it $(d,s)_{\rm D}$ – $2s$ in pairs degenerate Majorana neutrinos would constitute s massive Dirac ones, the rest being massless. Hence there would be $s(2d-s-1)/2$ mixing angles and $s(2d-s-1)/2 - d + 1$ phases [7]. It follows in particular that at $s = d \equiv n$ for this reduced type of $(n,n)_{\rm r}$ extension one would get $2n$ masses, $n(n-1)/2$ mixing angles and $(n-1)(n-2)/2$ phases in complete analogy to the quark sector.

The above results are not applicable at $s > d > 0$. Here the number of massive Dirac neutrinos saturates the maximum allowed value d, the rest of the $s - d$ Weyl neutrinos being massless. Hence the residual symmetry would increase up to $H = U(s-d) \times U(1)$, so that the number of mixing angles would be $d(d-1)/2$ and the number of phases $(d-1)(d-2)/2$. The results are summarized in Table 2 along with the case $(n,n)_{\rm D}$ for n complete families. It can be seen in particular that at fixed d the numbers of mixing angles and phases do not increase with the growth of s starting from $s = d - 1$.

3 Non-renormalizable extensions

3.1 Arbitrary case

Let us now generalize the preceding considerations to the most exhaustive Dirac–Majorana case with left-handed Majorana masses. The direct Majorana mass term for the doublet neutrinos is excluded in the minimal SM by the symmetry and renormalizability requirements. But in the extended SM as a low energy effective theory, it could stem from the SM invariant operator of the fifth dimension:

$$-\varDelta\mathcal{L} = \frac{1}{2\varLambda}(\phi^{C\dagger}\tau_i\phi)(\overline{l^C_{\rm R}} h {\rm i}\tau_2\tau_i l_{\rm L}) + {\rm h.}c., \tag{2}$$

with τ_i, $i = 1,2,3$ being the Pauli matrices, h being a $d \times d$ symmetric constant matrix, $\varLambda \gg v$ being the lepton number violating mass scale (supposedly of the order of the singlet Majorana masses) and v being the Higgs vacuum expectation value. The above operator with the effective isotriplet field $\varDelta_i = (1/\varLambda)(\phi^{C\dagger}\tau_i\phi)$ reflects the oblique radiative corrections in the low energy Lagrangian produced by the physics beyond the SM. In the unitary gauge, it yields the following mass and Yukawa term:

$$-\varDelta\mathcal{L} = \frac{1}{2}\left(1 + \frac{H}{v}\right)^2 \overline{\nu^C_{\rm R}} \mu\, \nu_{\rm L} + {\rm h.}c., \tag{3}$$

with $\mu = hv^2/\varLambda$. If the isotriplet $\varDelta_i$ were to be considered as an elementary one in the framework of renormalizable extensions, it would change only the emerging Yukawa interactions, not affecting the mass and mixing matrices.

There is no non-trivial residual symmetry in this case either, $H = I$. As for the free parameters, the phenomenological inclusion of such a mass term increases the numbers of moduli and phases by $d(d+1)/2$ each. Of the extra moduli, d ones are the Majorana neutrino masses, the rest being physical mixing angles. Hence, the extension amounts to $d + s$ independent neutrino masses, $d(d + 2s - 1)/2$ physical mixing angles and the same number of phases [6]. Let us designate this general type of the SM extension as (d,s), whether $s \le d$ or $s > d$. The parameter counting for this non-renormalizable extension of the SM is summarized in Table 3.

A special case without singlet neutrinos, i.e., the $(d,0)$ extension, results in $d(d-1)/2$ mixing angles and the same number of phases [4]. Clearly, the CP violation in the lepton sector becomes possible for two or more families without singlet neutrinos at all. On the other hand, the (n,n) extension with n complete families brings in $2n$ massive Majorana neutrinos with $n(3n-1)/2$ mixing angles and

Table 3. Parameter counting for the SM non-renormalizable extension (d, s) with d doublet and s singlet neutrinos. The symmetries are the same as in Table 1

Couplings	Moduli	Phases
Y^e, Y^ν, M	$d^2 + ds + s(s+1)/2$	$d^2 + ds + s(s+1)/2$
μ	$+d(d+1)/2$	$+d(d+1)/2$
$\mathcal{M}_{\rm ph}(d, s)$	$d(d+2s-1)/2 + (2d+s)$	$d(d+2s-1)/2$
$\mathcal{M}_{\rm ph}(n, n)$	$n(3n-1)/2 + 3n$	$n(3n-1)/2$

Table 4. Parameter counting for the SM extension $(d, s)_{\rm M}$ with only Majorana masses for the neutrinos. The symmetries are the same as in Table 1

Couplings	Moduli	Phases
Y^e, μ, M	$d^2 + d(d+1)/2 + s(s+1)/2$	$d^2 + d(d+1)/2 + s(s+1)/2$
$\mathcal{M}_{\rm ph}(d, s)_{\rm M}$	$d(d-1)/2 + (2d+s)$	$d(d-1)/2$

an equal number of phases [5]. Hence, CP violation might take place here already for one complete family.

3.2 Only Majorana masses

Let us consider a peculiar case of the general extension above. In the absence of Yukawa couplings, $Y^\nu \equiv 0$, but at non-zero Majorana masses, both left- and right-handed, the residual symmetry is still trivial ($H = I$) as in the general case. But now the doublet and singlet neutrino sectors completely disentangle from each other. All the $d+s$ Majorana neutrinos acquire masses, and we end up with $d(d-1)/2$ mixing angles and the same number of phases for the doublet neutrinos, without any mixing for the singlet ones, whether $s \le d$ or $s > d$. Let us designate this case $(d, s)_{\rm M}$. The results are collected in Table 4. We stress that the numbers of physical mixing angles and phases do not here depend on s. This is because the right-handed neutrinos are sterile in the case at hand, and their mixing matrix can be chosen to be unity on neglecting any other interactions. As for doublet neutrinos, this case formally corresponds to that $(d, 0)_{\rm M}$, which in turn coincides with the general one, $(d, 0)$.

4 Remarks

We would like to clarify some discrepancies for the renormalizable $(d, s)_{\rm r}$ extension between our counting in the weak basis and the one in the mass basis [6]. In the mass basis, an explicit new feature of the $(d, s)_{\rm r}$ extension, compared to the (d, s) one, is the appearance of the additional symmetry $U(d-s)$, $d \ge s$, due to $d-s$ neutrinos being massless. As a result, it is stated in the paper referred to that the mixing matrix for the $(d, s)_{\rm r}$ extension could be obtained from the corresponding general matrix just by deleting in the latter $(d-s)^2$ spurious parameters corresponding to $U(d-s)$. We would like to remark that this procedure is not sufficient to fix the number of independent moduli, and it generally overestimates the number of actual ones.

To illustrate this point, we note that it would follow from the prescription [6], e.g., that at $d = s \equiv n$ both $(n, n)_{\rm r}$ and (n, n) extensions would have the same numbers of mixing angles, as well as phases, $n(3n-1)/2$, in addition to $3n$ masses. On the other hand, an arbitrary square complex matrix Y can be uniquely written as a unitary matrix times a positive-definite Hermitian one, and a complex symmetric matrix M can be uniquely decomposed in terms of a unitary matrix V and a positive-definite diagonal one, $M = V^{\rm T} M_{\rm diag} V$. This means that taking into account the global symmetry G we could start in the $(n, n)_{\rm r}$ extension by choosing from the very beginning the Yukawa matrices Y^e and Y^ν as positive-definite Hermitian matrices and M as a positive-definite diagonal one. As we have thus exhausted the whole symmetry G and there is no non-trivial residual subgroup H, this set of parameters is the independent physical one. It contains $n(n+1)+n$ moduli and $n(n-1)$ phases. This completely agrees with Table 1 and is clearly less compared to [6].

We trace the origin of the discrepancy between the countings to the constraint $\mu = 0$ in (3). In passing from the (d, s) extension to the $(d, s)_{\rm r}$ one, it restricts $d(d+1)/2$ phases and the same number of physical moduli. In this, d of the conditions on the moduli can serve to determine d induced Majorana masses in terms of the mixing angles. Altogether, this leaves s independent Majorana masses, sd mixing angles and $d(s-1)$ phases. At $0 < s \le d$, there are s induced non-zero masses, $d-s$ neutrinos necessarily remaining massless. As a consequence of the inborn masslessness for $d-s$ neutrinos, the stated constraint supersedes here those gained from the $U(d-s)$ symmetry. E.g., according to the prescription [6] the extension $(d, 1)_{\rm r}$ should superficially correspond to $2d-1$ mixing angles and d phases, but an explicit construction shows that there are actually just d mixing angles, all of them being independent, and no phases at all. Especially clearly the above constraint works at $s > d$ when there appear no massless neutrinos and there is nothing to delete by the related transformations. Nevertheless, the counting of parameters at $s > d$ for the $(d, s)_{\mathrm{r}}$ extension proves to be not the same as for the (d, s) one.

5 Conclusion

The parameter counting in the weak basis is complementary to that in the mass basis. It allows one to gain clear insight into the independent physical parameter content of the SM extensions, both renormalizable and non-renormalizable, with arbitrary numbers of the singlet and left-handed doublet neutrinos.

Acknowledgements. The author is grateful to V.V. Kabachenko and L.B. Okun for very useful discussions.

References

1. N. Cabibbo, Phys. Rev. Lett. **10**, 531 (1963); M. Kobayashi, T. Maskawa, Prog. Theor. Phys. **49**, 652 (1972); L. Maiani, Phys. Lett. B **62**, 183 (1976)
2. B. Pontecorvo, Zh. Exp. Teor. Fiz. **34**, 247 (1958) [Sov. Phys. JETP **7**, 172 (1958)]; ibid. **53**, 1717 (1967) [Sov. Phys. JETP **26**, 984 (1968)]; Z. Maki, M. Nakagawa, S. Sakata, Prog. Theor. Phys. **28**, 870 (1962); V. Gribov, B. Pontecorvo, Phys. Lett. B **28**, 493 (1969); S.M. Bilenky, B. Pontecorvo, Phys. Rep. C **41**, 225 (1978)
3. S.M. Bilenky et al., Summary of the Phenomenology Working Group at the Europhysics Neutrino Oscillation Workshop (NOW'98), 1998, Amsterdam, hep-ph/9906251
4. J. Schechter, J.W.F. Valle, Phys. Rev. D **23**, 1666 (1981); M. Doi, T. Kotani, H. Nishiura, K. Okuda, E. Takasugi, Phys. Lett. B **102**, 323 (1981); J. Bernabeu, P. Pascual, Nucl. Phys. B **228**, 21 (1983); P.J. O'Donnell, U. Sarkar, Phys. Rev. D **52**, 1720 (1995), hep-ph/9305338
5. I.Yu. Kobzarev, B.V. Martemyanov, L.B. Okun, M.G. Schepkin, Yad. Fiz. **32**, 1590 (1980) [Sov. J. Nucl. Phys. **32**, 823 (1980)]; S.M. Bilenky, J. Hosek, S.V. Petcov, Phys. Lett. B **94**, 495 (1980); V. Barger, P. Langacker, J.P. Leveille, S. Pakvasa, Phys. Rev. Lett. **45**, 692 (1980)
6. J. Schechter, J.W.F. Valle, Phys. Rev. D **22**, 2227 (1980)
7. J. Donoghue, Phys. Rev. D **18**, 1632 (1978); J. Schechter, J.W.F. Valle, Phys. Rev. D **21**, 309 (1980)
8. A. Santamaria, Phys. Lett. B **305**, 90 (1993), hep-ph/9302301
9. Yu.F. Pirogov, O.V. Zenin, Eur. Phys. J. C **10**, 629 (1999), hep-ph/9808396